HIGH-SPEED ASYMPTOTIC FORM OF THE WAVE RESISTANCE OF BODIES IN A UNIFORMLY STRATIFIED LIQUID

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Below we analyze the asymptotic form of the velocity (Froude-number) dependence of the radiation energy loss (wave resistance) of nonlocally distributed mass sources equivalent to moving bodies. For three-dimensional bodies moving rapidly in a boundless uniformly stratified liquid and in a waveguide layer of finite depth, the asymptotic forms are similar (wave resistance $R \sim \ln \mathrm{Fr}/\mathrm{Fr}^2$ and $R \sim 1/\mathrm{Fr}^2$, respectively). For small bodies the leading term of the asymptotic form depends only on the dipole moment of the distributions of the sources, which can be calculated within the framework of the theory of a homogeneous liquid and is proportional to the volume of the body.

The "paradox of infinite wave resistance" occurs for uniformly distributed sources, elongated in the direction of the motion. The wave number integral in the infinite problem and the mode series in the waveguide problem are logarithmically divergent because of the logarithmically large contribution of short waves (wave modes with high numbers).

In the two-dimensional problem for a boundless liquid the wave resistance at high velocities is inversely proportional to the Froude number, while in the case of a waveguide the number of excited modes drops as the velocity increases and the radiation vanishes altogether at supercritical velocities.

1. Horizontal Motion in a Boundless Liquid. If the perturbations caused by a body moving in a density-stratified incompressible heavy liquid can be considered to be small and the effect of the body can be replaced by the equivalent effect of a distribution of mass sources m, the perturbations with velocity $\mathbf{v}$, density ρ, and pressure p will be found from the system of linear equations

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla p = \rho \mathbf{g}, \quad \frac{\partial \rho}{\partial t} + \frac{N^2}{g^2}\mathbf{g}\mathbf{v} = 0, \quad \nabla \mathbf{v} = m,$$

which are written in the Boussinesq approximation, and the density of the unperturbed state with allowance for the buoyancy frequency N(z) is set equal to unity.

In the case of a boundless uniformly stratified (N = const) liquid the energy loss per unit path length (wave resistance) of mass sources in uniform horizontal motion is represented, in accordance with these equations, by an integral quadratic form with respect to the source [1, 2],

$$R = \frac{1}{v_0}\int d\mathbf{r}\, d\mathbf{r}'\, w(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r})\, m(\mathbf{r}'),$$
$$2\pi^2 v_0 w(\mathbf{r}) = \int\limits_0^N d\omega \sqrt{N^2-\omega^2}\cos\frac{\omega x}{v_0} F\left(\frac{\omega}{v_0}\sqrt{\left|y^2+z^2-\frac{N^2}{\omega^2}z^2\right|}\right). \tag{1.1}$$

The x axis here coincides with the direction of motion, the function $F(\xi)$ reduces to the cylindrical functions $K_0(\xi)$ at $y^2 + z^2 > N^2z^2/\omega^2$ and $-(\pi/2)Y_0(\xi)$ with the inequality reversed.

At high velocities the source of the kernel of the quadratic form can be written as the expansion

$$w(\mathbf{r}) = \frac{a_1 \ln v_0 + b_1}{v_0} + \frac{a_2 \ln v_0 + b_2}{v_0^3} + O\left(\frac{\ln v_0}{v_0^5}\right), \tag{1.2}$$

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whose lower coefficients are given by

$$a_1 = \frac{N^2}{8\pi}, \quad a_2 = \frac{N^4(y^2-2x^2-3z^2)}{128\pi}, \quad b_1 = -\frac{N^2}{8\pi}I_1,$$
$$b_2 = \frac{N^4}{128\pi}[4z^2 I_1 + (2x^2 - y^2 - z^2) I_2 + y^2 - 3z^2],$$
$$I_n(y, z) = \frac{4}{\pi}\int_0^{\pi/2} d\alpha \sin^2(n\alpha) \ln\left(\frac{\gamma N}{2}\sqrt{|z^2\sin^2\alpha - y^2\cos^2\alpha|}\right).$$

The velocity dependence of the wave resistance is determined in addition to the indicated velocity dependence of the kernel w(r) by the form of the distribution of mass sources. For distributions modeling a body immersed in an incompressible liquid, the total intensity of the sources, which is proportional to the mass flow, should become zero and, therefore, the term with a_1 drops out of the formula for the wave resistance. When bodies are modeled with distributions of the dipole type

$$\int dx\, m(\mathbf{r}) = 0, \quad d \equiv \int d\mathbf{r}\, x m(\mathbf{r}) \neq 0 \tag{1.3}$$

the expansion of the wave resistance in velocities, which corresponds to (1.2), becomes

$$R = \frac{N^4 d^2}{32\pi v_0^4}(\ln v_0 + C) + O\left(\frac{\ln v_0}{v_0^6}\right),$$
$$Cd^2 = -\int d\mathbf{r}\, d\mathbf{r}'\, xx'\, m(\mathbf{r})\, m(\mathbf{r}')\, I_2(y-y',\ z-z'). \tag{1.4}$$

It is remarkable that the main high-velocity contribution here is determined by a general characteristic of the distributions of sources equivalent to the body (partially unknown or difficult to find) such as the total dipole moment d. Even the next term of the expression, however, depends on much larger details of these distributions and, hence, the shape of the bodies.

Besides the implicit velocity dependence indicated above there also is an implicit velocity dependence of the shape of the mass sources. In a homogeneous incompressible liquid with a linear description the latter is also linear: $m \sim v_0/\ell_0$, $d = d_0 \sim v_0\ell_0{}^3$, ... ($\ell_0$ is the characteristic size of the body). A dimensionless parameter $\mathrm{Fr} = v_0/(N\ell_0)$ arises in a stratified liquid characterized by a buoyancy frequency N and the velocity dependence of the source is more involved: $d = v_0\ell_0{}^3 f(\mathrm{Fr}) = d_0 f(\mathrm{Fr})/f(\infty)$, ...

It is to be expected that at velocity expansions analogous to those discussed above, which in the limit $\mathrm{Fr} \to \infty$ lead to the result for a homogeneous liquid (in particular, $d \to d_0$), are valid for sources at high velocities. This expectation is supported by the example of a flow of a uniformly stratified liquid around a cylinder. According to [3, 4] the coefficients of an equivalent round cylinder of radius ℓ_0 of a multipole expansion of the mass source

$$m = -\left[d + q_1\frac{\partial}{\partial x} + q_2\left(3\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\right) + \ldots\right]\frac{\partial}{\partial x}\delta(\mathbf{r})$$

are given at high values of Fr by the estimates

$$\frac{d}{d_0} = 1 - \frac{1}{2\,\mathrm{Fr}^2}\left(\ln\frac{2\,\mathrm{Fr}}{\gamma} + \frac{1}{2}\right) + O\left(\frac{\ln \mathrm{Fr}}{\mathrm{Fr}^4}\right),$$
$$\frac{q_1}{d_0 l_0} = \frac{1}{12\,\mathrm{Fr}^3} + O\left(\frac{\ln \mathrm{Fr}}{\mathrm{Fr}^5}\right), \quad \frac{q_2}{d_0 l_0^2} = O\left(\frac{1}{\mathrm{Fr}^8}\right), \ldots,$$

so that at an accuracy $o(\ln \mathrm{Fr}/\mathrm{Fr}^2)$ the discussion can be confined to dipole sources with dipole moment d_0 characteristic of a homogeneous liquid. Moreover, it turns out [4] that such an approximation works fairly well up to $\mathrm{Fr} \approx 1$.

Leaving only the leading terms of the high-velocity asymptotic form of the wave resistance in Eq. (1.4), therefore, we can take the values of d and C in a homogeneous unstratified liquid, $d_0 = v_0 \ell_0^3 \Delta_0$ and C_0, to be the characteristics of the sources. We rewrite the formula for the wave resistance in terms of Δ_0, C_0, and Fr as

$$R \approx \frac{N^2 l_0^4 \Delta_0^2}{32\pi} \frac{\ln \mathrm{Fr} + C_0}{\mathrm{Fr}^2}, \quad \mathrm{Fr} \gg 1. \tag{1.5}$$

For a body of arbitrary shape the ratio $d_0/v_0 = \ell_0^3 \Delta_0$ consists of the volume of the body and the apparent volume (mass) [5]. In particular, for a sphere of radius ℓ_0 we obtain $d_0/v_0 = 4\pi\ell_0^3/3 + 2\pi\ell_0^3/3$, i.e., $\Delta_0 = 2\pi$.

The expression for C in terms of the mass sources is simplified if the discussion is confined to sources localized in the horizontal (z = 0) or vertical (y = 0) planes. Formula (1.4) with allowance for

$$I_2(\xi, 0) = I_2(0, \xi) = I_1(0, \xi) - \frac{1}{4} = I_1(\xi, 0) + \frac{3}{4} = \ln \frac{\gamma N |\xi|}{4} + \frac{1}{4}$$

in the case of plane horizontal distributions of dipole sources becomes

$$R = \frac{N^4}{32\pi v_0^4} \int dy\, dy' \left(\ln \frac{4v_0}{\gamma N |y - y'|} - \frac{1}{4} \right) D(y)\, D(y'), \tag{1.6}$$
$$D(y)\, \delta(z) = \int dx\, x m(\mathbf{r}).$$

Exactly the same formula with the exchange $y \rightleftarrows z$ is valid for sources smeared in the vertical plane. By their general form they resemble the well known Kármán formula for the wave resistance of an elongated body moving with supersonic velocity [6]. The mass sources here, however, are distributed over a plane and not only in one direction of motion. For one-dimensional longitudinal distributions, as is clear from (1.6) and the initial formula (1.1), the wave resistance becomes infinite. This paradox of an infinite wave resistance is due to the exaggerated role of short transverse internal waves, making logarithmically larger contributions to the resistance, with such modeling of bodies. These contributions are annihilated by interference when the transverse nonlocal nature of the modeling sources is taken into account.

2. Motion of a Body in a Layer of Finite Depth. The wave resistance of mass sources, moving uniformly horizontally in a horizontal waveguide between hard caps z = 0 and z = h, filled with a stratified liquid, is represented as the sum of independent contributions from various wave modes [1]:

$$R = \sum_n R_n = \frac{1}{v_0} \sum_n \int d\mathbf{r}\, d\mathbf{r}'\, w_n(x - x', y - y';\ z, z')\, m(\mathbf{r})\, m(\mathbf{r}'),$$
$$w_n(x, y;\ z, z') = \int_0^\infty dk \frac{\omega_n^4 H(\varkappa_n)}{2\pi k \sqrt{\varkappa_n}} \frac{\partial \psi_n}{\partial z_1} \frac{\partial \psi_n}{\partial z_2} \cos \frac{x\omega_n}{v_0} \cos \frac{y\sqrt{\varkappa_n}}{v_0}. \tag{2.1}$$

The integration over z is carried out here from 0 to h; $\varkappa_n \equiv k^2 v_0^2 - \omega_n^2$; $H(\varkappa_n)$ is the Heaviside function; and $\psi_n = \psi_n(k, z)$ and $\omega_n = \omega_n(k)$ are the eigenfunctions and eigenvalues of the boundary-value problem

$$\left[\frac{\partial^2}{\partial z^2} - k^2 + \frac{k^2}{\omega_n^2} N^2(z) \right] \psi_n(k, z) = 0, \quad \psi_n|_{z=0} = \psi_n|_{z=h} = 0.$$

For distributions of sources of the dipole type (1.3), equivalent to the immersed body, $w_n(0, y; z, z')$ does not make a contribution to R so that when estimating the resistance of bodies from Eq. (2.1), we must replace w_n by $\delta w_n \equiv w_n(x, y; z, z') - w_n(0, y; z, z')$. In the case of high velocities ($v_0 \gg \ell_0 N_{max}$, where $\omega_n < N_{max} < \infty$) in the first approximation we have $\delta w_n \approx \partial^2 w_n/\partial x^2|_{x=0} x^2/2$ and the formula for the wave resistance could be rewritten as [see Eq. (1.6)]

$$R = -\frac{1}{v_0}\sum_n \int dy\,dy' \int_0^h dz\,dz' \left.\frac{\partial^2 w_n}{\partial x^2}\right|_{x=0} D(y, z)\,D(y', z'),$$
$$D(y, z) = \int dx\, xm(\mathbf{r}). \tag{2.2}$$

The solution of the problem for the eigenvalues for a uniformly stratified liquid is simple:

$$\omega_n^2 = k^2 c_n^2 = \frac{N^2 k^2}{k^2 + (\pi n/h)^2}, \quad \psi_n = \sqrt{\frac{2}{k^2 N^2 h}} \sin\frac{\pi n z}{h},$$

finding the kernel in the resistance formula (2.2) comes down to calculating the integral of the elementary functions

$$-\left.\frac{\partial^2 w_n}{\partial x^2}\right|_{x=0} = \frac{N^4}{\pi^2 v_0^3 n} \cos\frac{\pi n z}{h} \cos\frac{\pi n z'}{h} I\left(\frac{\pi n\,|y - y'|}{h}, \frac{Nh}{\pi n v_0}\right),$$
$$I(\xi, \eta) \equiv \int_0^\infty dq \frac{q^2 H(1 + q^2 - \eta^2)}{\sqrt{(1+q^2)^5 (1 + q^2 - \eta^2)}} \cos\left(\xi q \sqrt{\frac{1 + q^2 - \eta^2}{1 + q^2}}\right). \tag{2.3}$$

The latter is particularly simple at $\eta \ll 1$:

$$I(\xi, 0) = (\pi/16)(1 + |\xi| - \xi^2) \exp(-|\xi|), \tag{2.4}$$

which makes it possible to write a formula for the wave resistance of bodies moving in a waveguide with supercritical velocities $v_0 \gg \frac{Nh}{\pi} = c_{10} = \lim_{k\to 0} c_1 \geqslant c_1 > c_n$, i.e., faster than all the internal waves, as

$$R = \frac{N^4}{\pi^2 v_0^4} \sum_n \frac{1}{n} \int dy\,dy'\, I\left(\frac{n\pi}{h}|y - y'|,\ 0\right) M_n(y)\,M_n(y'),$$
$$M_n(y) \equiv \int_0^h dz \cos\frac{\pi n z}{h} \int dx\, xm(\mathbf{r}), \tag{2.5}$$

which gives the asymptotic dependence of the resistance in the waveguide on the Froude number (at Fr $\gg$ 1) of the power-law type $R \sim 1/\mathrm{Fr}^2$ (the discussion is similar to Sec. 1).

Compared to the case in the analogous asymptotic form for a boundless liquid [see Eqs. (1.6), (2.4), and (2.5)], the leading term of the high-velocity asymptotic form of the resistance of bodies in a waveguide is much more sensitive to particular features of the equivalent distributions of mass sources and, hence, the shapes of the bodies.

The fact that as the depth of the waveguide increases ($h \to \infty$) the high-velocity asymptotic form $R \sim 1/\mathrm{Fr}^2$ does not go over into the asymptotic form $R \sim \ln \mathrm{Fr}/\mathrm{Fr}^2$ for a boundless liquid is not surprising. The asymptotic form for a waveguide is "intermediate" since it was obtained on the assumption that $v_0\pi \gg hN$.

From (2.3)-(2.5) we can see that the higher wave modes with $n \gg n_0 = [h/(\pi\ell_0)]$ for distributions of sources that are transversely nonlocal make a small contribution to the high-velocity asymptotic form ($v_0 \gg c_{10}$) of the resistance. The contribution is suppressed exponentially [see Eq. (2.4)] along the horizontal and because of rapid oscillations [see Eq. (2.5)], along the vertical.

The result (2.5) can be simplified substantially for lower modes with $n \ll n_0$ (it is understood that $n_0 \gg 1$) and the leading term of the high-velocity asymptotic form for bodies that are small in comparison with the waveguide depth ($\ell_0 \ll h$) depends only on the dipole moment of the source distribution. This conclusion, we must bear in mind, is valid only outside the neighborhood of zeros of the functions cos $(\pi n z_0/h)$ (z_0 is the coordinate of the horizon of the motion of the symmetry center of the body), since the simplification is based on the approximate estimate

$$\int dy\, M_n(y) = \int d\mathbf{r}\, xm(\mathbf{r}) \cos\frac{\pi nz}{h} \approx d\cos\frac{\pi nz_0}{h},$$

which requires refinement near these zeros.

Finally, the formula for the wave resistance can be limited, with acceptable accuracy, to a finite number of terms with the particularly simple (with the above proviso) form of the lowest terms

$$R = \sum_{n \leqslant n_0} R_n, \quad R_n \approx \frac{N^4 d^2}{16\pi v_0^4} \frac{1}{n} \cos^2\frac{\pi nz_0}{h} \quad (n \ll n_0). \tag{2.6}$$

If we use the estimate mentioned above for all the terms of the sum (and not only at $n \ll n_0$), we arrive at an approximate formula for the resistance in a waveguide:

$$R \approx \frac{N^4 d^2}{32\pi v_0^4} \ln\left(\frac{\gamma n_0}{2} \Big/ \left|\sin\frac{\pi nz_0}{h}\right|\right) \approx \frac{N^2 l_0^4 \Delta_0^2}{32\pi\,\mathrm{Fr}^2} \ln\frac{h}{l_0},$$
$$\mathrm{Fr} \gg h/l_0 \gg 1.$$

An entirely different formula is obtained if we ignore the transverse nonlocality of the sources modeling the body. At $\xi = 0$ the integral from (2.3) and (2.5) reduces to complete elliptic integrals

$$3\eta^4 I(0, \eta) = \begin{cases} \eta(\eta^2 - 1) K\left(\frac{1}{\eta}\right) - \eta(\eta^2 - 2) E\left(\frac{1}{\eta}\right), & \eta > 1, \\ 2(\eta^2 - 1) K(\eta) - (\eta^2 - 2) E(\eta), & \eta < 1 \end{cases}$$

and becomes independent of the number of the constant $I(0, 0) = \pi/16$ at $\eta = Nh/(\pi v_0 n) \ll 1$, i.e., for modes with high numbers. The wave resistance [see (2.5)] for one-dimensional longitudinal distributions of sources, therefore, is a series in the modes that is logarithmically divergent at high numbers [see (2.6)]:

$$R \approx \frac{N^4 d^2}{16\pi v_0^4} \sum_{n=1}^{\infty} \frac{1}{n} \cos^2\frac{\pi nz_0}{h}.$$

This is analogous to the situation in a boundless liquid. The vertical component k_z of the wave vector is quantized ($k_z = \pi n/h$) in a waveguide layer of a uniformly stratified liquid of finite thickness. A logarithmically large contribution from high modes, therefore, corresponds here to the logarithmically large contribution with large vertical components of the wave vectors in a boundless liquid (Sec. 1). In both cases the "paradox of an infinite wave resistance" is due to the exaggeration of the contribution of short transverse waves in the modeling of bodies with one-dimensional longitudinal distributions.

3. Two-Dimensional Problem. The differences between a boundless liquid and a waveguide of finite depth manifest themselves even more clearly in the two-dimensional problem. The formula for the wave resistance in a boundless liquid can now be written as an integral quadratic form, similar to (1.1), with the following kernel of $w(x, z)$:

$$\frac{\partial w}{\partial x} = -\frac{1}{2\pi v_0} \int_0^N d\omega \sqrt{N^2 - \omega^2} \sin\frac{\omega x}{v_0} \cos\frac{z\sqrt{N^2 - \omega^2}}{v_0}.$$

For source distributions of the type (1.3) at high Froude numbers ($\mathrm{Fr} = v_0/(N\ell_0) \gg 1$) the leading term of the asymptotic form depends only on their dipole moment: accordingly, with the same accuracy we can use the dipole moment from the theory of a homogeneous liquid:

$$R \approx \frac{N^3 d^2}{6\pi v_0^3} \approx \frac{N^2 l_0^3 \Delta_0^2}{6\pi\,\mathrm{Fr}}, \quad \mathrm{Fr} \gg 1. \tag{3.1}$$

Here $\Delta_0 = d_0/(v_0 \ell_0{}^2)$ is the dimensionless dipole moment of the distribution of mass sources, equivalent to a body with a homogeneous liquid flowing around it. It is 2π for a round cylinder of radius ℓ_0.

In the case of transverse horizontal motion of a cylindrical body in a waveguide layer of finite depth the wave resistance of the distribution of mass sources equivalent to the body is the sum of independent contributions of a finite number of modes [1]:

$$R = \sum_n R_n = \frac{1}{v_0} \sum_n \int dx\,dx' \int_0^h dz\,dz'\, w_n(x - x';\ z,\ z')\, m(x,\ z)\, m(x',\ z'),$$

$$w_n(x;\ z,\ z') = \frac{v_0^5}{2} \left| \frac{\partial c_n^2}{\partial k^2} \right|_{k=k_n}^{-1} \cos k_n x\, \frac{\partial \psi_n(k_n,\ z)}{\partial z} \frac{\partial \psi_n(k_n,\ z')}{\partial z'} H(c_{n0} - v_0)$$

(k_n is the unique root of $c_n(k) = v_0$ with fixed number). From the formula for w_n it is clear that the series in the modes breaks off at the term for which $c_{n+1,0} < v_0 < c_{n0}$. Since the velocities of the internal waves decrease monotonically as the mode number increases, the number of excited waveguide modes obviously decreases as the velocity grows.

In the case of a uniformly stratified (N = const) liquid the wave resistance of bodies modeled by distributions of the type (1.3) is written, as high Froude numbers ($\mathrm{Fr} = v_0/(N\ell_0) \gg 1$) in the simplified form

$$R = \frac{\pi^2}{h^3} \sum_{n=1}^{n_m} n^2 M_n^2 H(c_{n0} - v_0),$$

$$M_n = \int_0^h dz \cos\frac{\pi n z}{h} \int dx\, x m(x,\ z), \quad n_m \equiv \left[\frac{Nh}{\pi v_0}\right].$$

Since the condition for the emission of waves in the two-dimensional problem implies that $n_m \geq 1$, with this accuracy we can go over from M_n to the dipole moment d ($M_n \approx d \cos \pi n z_0/h$) with the same proviso concerning the nodal points as in Sec. 2. The sum over the modes is then easily calculated:

$$R = \frac{\pi^2 d^2}{12 h^3} \left\{ n_m(n_m + 1)(2n_m + 1) - \frac{3}{4} \frac{\partial^2}{\partial \zeta^2} \frac{\sin(2n_m + 1)\zeta}{\sin \zeta} \right\},$$
$$n_m = \left[\frac{Nh}{\pi v_0}\right] \geqslant 1, \quad \zeta \equiv \frac{\pi z_0}{h}. \tag{3.2}$$

Here the asymptotic formula has been obtained for an arbitrary distribution of sources of the dipole type (1.3) in the limit of high values of Fr. It coincides with the exact formula that holds for a point dipole for any Fr [1].

At $n_m \gg 1$ the reistance formula (3.2) for a waveguide goes over into formula (3.1) for a boundless liquid. This happens because the conditions $Nh/\pi \gg v_0 \gg N\ell_0$ are satisfied here. As the velocity of the body increases the number of excited modes in the two-dimensional problem drops (only one mode is excited at $Nh > \pi v_0 > Nh/2$) and at supercritical velocities (in a uniformly stratified liquid at $v_0 > c_{10} = Nh/\pi$) radiation does not occur. Therein lies the distinctive feature of the two-dimensional waveguide problem.

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ONSET OF COHERENT LARGE-SCALE MOTION IN A PLANE TURBULENT WAKE

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In the present article we give the results of a theoretical investigation of the response of a plane (two-dimensional) turbulent wake to an external harmonic disturbance. The underlying concepts and the approach used for the stated problem are discussed in [1]. Apart from the fact that the flow geometry differs from [1], we also consider the influence of crossflow variation of the turbulent viscosity on the evolution of large-scale disturbances.

1. Self-Similar Plane Turbulent Wake. Following [1], we introduce the turbulent Reynolds number for a self-similar wake:

$$\mathrm{Re}_T = u_0 b/\nu_T (\equiv \mathrm{const}), \tag{1.1}$$

where $\nu_T(X) \sim u_0 b$ is a characteristic turbulent viscosity in the cross section of the wake at the longitudinal coordinate $X = (x - x_0)$ measured from a fictitious origin x_0, and u_0 and b are local velocity and length scales. The length scale is given by the relation

$$b = (\nu_T X/U_\infty)^{1/2}. \tag{1.2}$$

The resistance offered by the body against a flow with velocity U_∞ has the form

$$F = \rho \int\limits_{-\infty}^{\infty} U(U - U_\infty)\,dy \quad (\equiv \rho U_\infty^2 \theta) \tag{1.3}$$

(θ is the momentum loss thickness). We represent the average flow velocity in self-similar far-wakes by the expression

$$U = U_\infty[1 - \varepsilon\varphi_0(\eta)],\ V = U_\infty \varepsilon^2 \chi_0(\eta) \tag{1.4}$$

($\varepsilon = u_0/U_\infty \ll 1$ and $\eta = y/b$ is the dimensionless transverse coordinate). We rewrite Eq. (1.3):

$$\theta = \varepsilon b J_1 - \varepsilon^2 b J_2,\ J_n = \int\limits_{-\infty}^{\infty} \varphi_0^n(\eta)\,d\eta, \quad n = 1,\ 2. \tag{1.5}$$

Disregarding the term $O(\varepsilon^2)$ in Eq. (1.5) and making use of Eqs. (1.1) and (1.2), we obtain expressions for the local scales:

$$u_0/U_\infty = C(X/\mathrm{Re}_T)^{-1/2},\ b = C(X/\mathrm{Re}_T)^{1/2},\ C = (\theta/J_1)^{1/2}. \tag{1.6}$$

We express Re_T in a form suitable for experimental evaluation:

$$\mathrm{Re}_T = (Xu_0)/(bU_\infty). \tag{1.7}$$

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